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An amplifying solution of the Maxwell–Bloch equations with atomic relaxation and field losses

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Abstract. Linear analytic solutions of the homogeneously broadened, unidirectional Maxwell–Bloch equations with field losses are found in circumstances where amplification can occur. Particular stress is laid upon the similarity-variable solution. A number of inequalitiesure derived which suffice to differentiate between the physical regimes of uperfluorescence, amplified spontaneous emission and swept-gain amplification.

1. Introduction

The emission of radiant energy from a system consisting of a large number of inverted atoms can take place in a variety of different ways. The precise mechanism of an emission process is determined largely by the relative magnitudes of the characteristic time constants of the system, these being associated with the interplay of the various gain and loss processes. A set of unidirectional (reduced) Maxwell–Bloch equations (MBE), which describes successfully the interaction of plane-wave, single-mode radiation with an array of two-level atoms, forms the common basis for the theoretical treatment of a large catalogue of nonlinear optical phenomena, some important examples of recent interest being superfluorescence (SF), amplified spontaneous emission (ASE), steady-state processes in amplifiers including swept-gain amplification (SGA), and also stimulated Raman scattering, resonant multiphoton processes, solitary wave behaviour, and in absorbing media the hysteresis and bistability phenomena.

In practice, a variety of amplifier problems are found to be characterised by field and atomic polarisation envelopes which depend upon a 'similarity' coordinate (a variable consisting of the product of a spatial and a temporal coordinate, suitably scaled). Following the work of Lamb (1971), it is clear that pi pulses should propagate in a lossless amplifier with 'ringing' field profiles, which are obtained as a similarity-variable solution of the equations of motion. Burnham and Chiao (1969) also found ringing-pulse solutions of this type in their analysis of induced, coherent resonance fluorescence from a thin slab of absorbing material. It would therefore seem profitable to investigate the similarity solution of the reduced MBE in as general a form as possible. The introduction of dissipation processes into the physics undoubtedly complicates the analysis, but is unavoidable if a realistic theoretical description is desired, and indeed is an *essential* ingredient in the dynamics of ASE and SGA.

We focus our attention upon a linearised solution of the MBE, it being reasonable to expect that this may be utilised as a basis for attacking the much more difficult problem of the nonlinear regime. Fortunately the linear solution is of considerable interest in its

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own right, and it should be mentioned that in SF the behaviour of the radiating system in the linear regime strongly affects the subsequent behaviour in the nonlinear regime. The shapes and delays of the pulses emitted in SF are strongly affected by the effective initial 'tipping angle' of the collective Bloch vector, which is a measure of the initiating effect of noise. Note also that, although the approach adopted here is semiclassical, with phenomenological treatment of the dissipative processes, the solutions to the equations of motion can be regarded as both prerequisites and guides for fully quantum-mechanical treatments. For this reason, and because semiclassical descriptions are often good approximations anyway, the present use of semiclassical formalism can be justified. Fully quantum-mechanical treatments of SF in the linear regime have recently been explored by Glauber and Haake (1978) and by Schuurmans and Polder (1979), and yield expressions for the fields and atomic polarisation envelopes which are similar to the semiclassical counterparts to be derived below. The latter quantities will be regarded as real, since only on-resonant and unchirped processes have been assumed.

2. The equations of motion

For the purposes of this investigation the equations of motion describing the interaction between a rod-shaped array of two-level atoms of length L and an electromagnetic field propagating along the axis of the rod are assumed to be

$$\frac{\partial \mathfrak{E}}{\partial x}+\frac{1}{c}\frac{\partial \mathfrak{E}}{\partial t}+k\mathfrak{E}=c^{-1}\beta P \tag{1a}$$

$$\frac{\partial P}{\partial t}+T_2^{-1}P=\mathfrak{E}N \tag{1b}$$

$$\frac{\partial N}{\partial t}+T_1^{-1}(N+N_0)=-\mathfrak{E}P \tag{1c}$$

where $\mathfrak{E}(x,t)$ is the electric-field envelope, $P(x,t)$ is the envelope of the atomic polarisation density and $N(x,t)$ is the envelope of the population-inversion density (all slowly varying quantities), while T_1 and T_2 are longitudinal and transverse decay times, k is a constant linear field loss coefficient with the dimension of inverse length and $\beta=2\pi\omega_0\mu^2\rho/\hbar$ (ω_0 is the on-resonance frequency, μ is the dipole matrix element and ρ is the density of active atoms). For an amplifying medium the parameter β is taken by convention (Lamb 1971) to be positive and is related to the 'superradiance time' T_R (Bullough *et al* 1978) through

$$\beta=2T_R^{-1}T_E^{-1} \tag{2}$$

where $T_E=2L/c$ is the 'round-trip' time. It is assumed that the medium is inverted by a swept δ-pulse excitation, and only emission of energy along the direction of propagation of the excitation is considered.

Before proceeding with the analysis, it is worth recording that a number of different scalings of the time and space variables, x and t, as well as of the field variable $\mathfrak{E}(x,t)$, have been reported in the literature dealing with SF and ASE. Some of the recent treatments of SF theory are both quantum-mechanical and bidirectional (i.e. include at least two coupled, counterpropagating fields); these important features do not affect the types of scaling procedures, however, and will be omitted from the present discussion. The works of Bullough *et al* (Saunders and Bullough 1977, Bullough *et al* 1978) and the present author (Hermann 1979a, b) employ T_E as a scaling parameter, i.e. with $E=\frac{1}{2}T_E\mathfrak{E}$, $\bar{T}_i=\gamma_i^{-1}=(\frac{1}{2}T_E)^{-1}T_i$ and $x'=x/L$, $t'=ct/L$ we can re-express equations (1*a*)–(1*c*) in terms of the displaced-time frame

$$\tau=t'-x' \qquad \mu=x' \tag{3}$$

in the slightly simpler form

$$\frac{\partial E}{\partial \mu}+kLE=\alpha P \tag{4a}$$

$$\frac{\partial P}{\partial \tau}+\gamma_2 P=EN \tag{4b}$$

$$\frac{\partial N}{\partial \tau}+\gamma_1(N+N_0)=-EP \tag{4c}$$

with $\alpha=\frac{1}{2}T_E/T_R=(\frac{1}{2}T_E/T_c)^2$; $T_c=\beta^{-1/2}$ is the Arecchi–Courtens (1970) 'maximum cooperation time', the 'cooperation length' being cT_c.

An alternative procedure, which has been adopted by Glauber and Haake (1978) and by Haake *et al* (1979), is to scale the time variable to T_R. Thus, with $T=t/T_R$, $X=x'=x/L$, $T_i'=T_i/T_R$ and $E=\frac{1}{2}T_E\mathfrak{E}$ we have

$$\frac{\partial E}{\partial T}+\alpha^{-1}\left(\frac{\partial E}{\partial X}+kLE\right)=P \tag{5a}$$

$$\frac{\partial P}{\partial T}+T_2'^{-1}P=\alpha^{-1}EN \tag{5b}$$

$$\frac{\partial N}{\partial T}+T_1'^{-1}(N+N_0)=-\alpha^{-1}EP. \tag{5c}$$

Schuurmans and Polder (1979) have scaled *both* time and field variables to T_R; i.e. with $E'=T_R\mathfrak{E}$, $T_i'=T_i/T_R=\gamma_i'^{-1}$, $T'=(t-x/c)/T_R=\alpha(t'-x')$, $X'=x'$ we find that equations (1*a*)–(1*c*) transform to

$$\frac{\partial E'}{\partial X'}+kLE'=P \tag{6a}$$

$$\frac{\partial P}{\partial T'}+\gamma_2'P=E'N \tag{6b}$$

$$\frac{\partial N}{\partial T'}+\gamma_1'(N+N_0)=-E'P. \tag{6c}$$

The forms of the MBE given by sets (5) and (6) are probably more convenient than the set (4) for computational purposes, provided that superradiance occurs. In a more general analytical treatment, however, the set (4) is preferable since it is possible to relate the mathematical expressions to the physics without transforming back to the original variables. Consequently, we will use the set (4) as the basis for our analysis hereafter, and will reserve the use of the Greek letters τ and μ for time and space coordinates which have been scaled in a dimensionless form.

3. The general linear solution

The linearisation of the set of equations (4a)–(4c) within the context of an amplifying process corresponds with the assumption that $N \simeq 1$ at all positions and times. If the dephasing (characteristic time γ_2^{-1}) and the field losses (characteristic time $c^{-1}k^{-1}$) are the dominant dissipative processes, then equation (4c) may be ignored. It is then possible to separate equations (4a) and (4b) into equations for $P(\mu, \tau)$ and $E(\mu, \tau)$, and this may be accomplished in two ways. The first way is to apply formal integration procedures to equations (4a) and (4b), with $N(\mu, \tau) = 1$, from which it immediately follows that $E(\mu, \tau)$ and $P(\mu, \tau)$ are respective solutions of the pair of integral equations

$$E(\mu, \tau) = E(0, \tau)\,\mathrm{e}^{-\kappa\mu} + \alpha\,\mathrm{e}^{-\gamma\tau}\int_0^\mu P(\mu', 0)\,\mathrm{e}^{\kappa(\mu'-\mu)}\,\mathrm{d}\mu' + \alpha\int_0^\mu\int_0^\tau E(\mu', \tau')\,\mathrm{e}^{\kappa(\mu'-\mu)+\gamma(\tau'-\tau)}\,\mathrm{d}\tau'\,\mathrm{d}\mu' \tag{7a}$$

$$P(\mu, \tau) = P(\mu, 0)\,\mathrm{e}^{-\gamma\tau} + \mathrm{e}^{-\kappa\mu}\int_0^\tau E(0, \tau')\,\mathrm{e}^{\gamma(\tau'-\tau)}\,\mathrm{d}\tau' + \alpha\int_0^\tau\int_0^\mu P(\mu', \tau')\,\mathrm{e}^{\kappa(\mu'-\mu)+\gamma(\tau'-\tau)}\,\mathrm{d}\mu'\,\mathrm{d}\tau' \tag{7b}$$

where we have written κ for kL.

The second way of performing the separation is to differentiate equation (4a) with respect to τ and equation (4b) with respect to μ. After some rearrangement, a separation into a pair of partial differential equations in $E(\mu, \tau)$ and $P(\mu, \tau)$ is achieved:

$$\frac{\partial^2 P}{\partial\mu\,\partial\tau} + \gamma\frac{\partial P}{\partial\mu} + \kappa\frac{\partial P}{\partial\tau} = (\alpha - \kappa\gamma)P \tag{8a}$$

$$\frac{\partial^2 E}{\partial\mu\,\partial\tau} + \gamma\frac{\partial E}{\partial\mu} + \kappa\frac{\partial E}{\partial\tau} = (\alpha - \kappa\gamma)E. \tag{8b}$$

These differential equations may be solved in a straightforward manner by the method of Laplace transforms. No specific assumptions concerning the initial or boundary conditions need be made at this stage. In the space defined by the transformation

$$\bar{P}(\mu, p) = \mathscr{L}\{P(\mu, \tau)\} \tag{9}$$

we find that equation (8a) becomes

$$\bar{P}(\mu, p) = \bar{P}(\mu = 0, p)\exp\left[\left(\frac{\alpha}{p+\gamma} - \kappa\right)\mu\right] + \int_0^\mu \frac{\mathscr{P}(\mu', 0)}{(p+\gamma)}\exp\left[-\left(\frac{\alpha}{p+\gamma} - \kappa\right)(\mu' - \mu)\right]\mathrm{d}\mu' \tag{10}$$

where $\mathcal{P}(\mu, 0) = \kappa P(\mu, \tau = 0) + \partial P(\mu, \tau = 0)/\partial\mu$. This equation may now be Laplace-inverted using the convolution theorem. After some manipulation we obtain the final result

$$P(\mu,\tau) = P(0,0)\,\mathrm{e}^{-(\gamma\tau+\kappa\mu)} I_0[2(\alpha\mu\tau)^{1/2}] + \mathrm{e}^{-\gamma\tau}\int_0^{\mu} \mathcal{P}(\mu',0)\,\mathrm{e}^{-\kappa(\mu-\mu')} I_0\{2[\alpha\tau(\mu-\mu')]^{1/2}\}\,\mathrm{d}\mu' + \mathrm{e}^{-\kappa\mu}\int_0^{\tau} E(0,\tau')\,\mathrm{e}^{-\gamma(\tau-\tau')} I_0\{2[\alpha\mu(\tau-\tau')]^{1/2}\}\,\mathrm{d}\tau' \tag{11a}$$

where $I_0(\theta)$ is a modified Bessel function of order zero. Equation (8*b*) can be solved similarly, and we find

$$E(\mu,\tau) = E(0,0)\,\mathrm{e}^{-(\gamma\tau+\kappa\mu)} I_0[2(\alpha\mu\tau)^{1/2}] + \alpha\,\mathrm{e}^{-\gamma\tau}\int_0^{\mu} P(\mu',0)\,\mathrm{e}^{-\kappa(\mu-\mu')} I_0\{2[\alpha\tau(\mu-\mu')]^{1/2}\}\,\mathrm{d}\mu' + \mathrm{e}^{-\kappa\mu}\int_0^{\tau} \mathcal{E}(0,\tau')\,\mathrm{e}^{-\gamma(\tau-\tau')} I_0\{2[\alpha\mu(\tau-\tau')]^{1/2}\}\,\mathrm{d}\tau' \tag{11b}$$

where $\mathcal{E}(0,\tau') = \gamma E(0,\tau') + \partial E(0,\tau')/\partial\tau'$. Note that equations (11*a*) and (11*b*) are both symmetric with respect to μ and τ, since $E(0,\tau) = \gamma P(0,\tau) + \partial P(0,\tau)/\partial\tau$ and $\alpha P(\mu,0) = \kappa E(\mu,0) + \partial E(\mu,0)/\partial\mu$.

Equations (11*a*) and (11*b*) are the general solutions of equations (4*a*) and (4*b*) in the linear regime, and do not appear to have been given in the literature in their present form previously. In the course of investigating a pulse propagation problem, Lax (1978) has given an analytic solution of a set of two coupled linear equations of first order and with arbitrary coefficients. Like equations (11*a*) and (11*b*), the solutions for the two dependent variables are displayed in integral form; however, they contain both I_0 and I_1 modified Bessel functions in the integrands. Straightforward integration by parts converts Lax's solutions into the forms of (11*a*) and (11*b*), which have more immediate application to the present amplifier problem.

Although they appear complicated at first sight, it happens that the integrals can sometimes be evaluated in terms of functions whose properties are well understood and tabulated. In order to accomplish this, it will be helpful to establish two mathematical identities. The form for the similarity variable that is found to be most convenient is

$$\theta(\mu,\tau) = 2(\alpha\tau\mu)^{1/2}, \tag{12}$$

from which we immediately deduce the differential property

$$\alpha^{-1}\frac{\partial^2}{\partial\mu\,\partial\tau} = \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \theta^{-1}\frac{\mathrm{d}}{\mathrm{d}\theta}. \tag{13}$$

The differential equation for a modified Bessel function of order zero, $y = I_0(\theta)$, is $y'' + \theta^{-1}y = y$. Using equation (13) we therefore deduce the identities

$$2\alpha\tau\theta^{-1}I_1(\theta) = \frac{\partial}{\partial\mu}I_0(\theta) = \alpha\int_0^{\tau} I_0(\theta')\,\mathrm{d}\tau' \qquad \theta' = 2(\alpha\mu\tau')^{1/2} \tag{14a}$$

$$2\alpha\mu\theta^{-1}I_1(\theta) = \frac{\partial}{\partial\tau}I_0(\theta) = \alpha\int_0^{\mu} I_0(\theta')\,\mathrm{d}\mu' \qquad \theta' = 2(\alpha\mu'\tau)^{1/2}. \tag{14b}$$

Considering the problem of the initiation of an emission process, it has been recognised (Schuurmans and Polder 1979) that a distinction may be drawn between theories which ascribe the initiation of SF to the uncertainty in the atomic polarisation at $\tau = 0$ (Haake *et al* 1979, Glauber and Haake 1978) and theories which ascribe the initiation to zero-point fluctuations of the incident vacuum field at $\mu = 0$ (Polder *et al* 1979, Schuurmans *et al* 1978). In common with Schuurmans and Polder (1979), we will allow for fluctuations of both field and matter (in the context of a semiclassical description) by introducing the initial and boundary conditions $P(\mu, \tau = 0) = f(\mu)$, $E(\mu = 0, \tau) = g(\tau)$, which in the simplest model are taken to be constants, P_0 and E_0 say. More generally, conditions such as these can be implemented readily and controlled at a macroscopic level, so that in adopting them it should be understood that we are not confining the theory solely to a simulation of the initiating effects of quantum processes alone. Using constant values P_0 and E_0, the equations (7*a*) and (7*b*) may be solved, giving in particular

$$P(0, \tau) = P_0\,\mathrm{e}^{-\gamma\tau} + \gamma^{-1}E_0(1-\mathrm{e}^{-\gamma\tau}) \tag{15a}$$

$$E(\mu, 0) = E_0\,\mathrm{e}^{-\kappa\mu} + (\alpha/\kappa)P_0(1-\mathrm{e}^{-\kappa\mu}). \tag{15b}$$

For a situation in which cooperative processes are totally absent, we may put $\alpha = 0$ in equations (11*a*) and (11*b*), so that a monochromatic EM field propagating in the axial direction will experience decay of its amplitude according to the time-independent law

$$E(\mu, \tau) = E_0\,\mathrm{e}^{-\kappa\mu}, \tag{16a}$$

while the atomic polarisation envelope will correspondingly decay according to

$$P(\mu, \tau) = P_0\,\mathrm{e}^{-\gamma\tau} + \gamma^{-1}E_0\,\mathrm{e}^{-\kappa\mu}(1-\mathrm{e}^{-\gamma\tau}). \tag{16b}$$

More general expressions for $E(\mu, \tau)$ and $P(\mu, \tau)$ when $\alpha > 0$ are obtained when equations (11*a*) and (11*b*) are evaluated with the help of identities (14*a*) and (14*b*). Again we assume constant values, P_0 and E_0, for $P(\mu, 0)$ and $E(0, \tau)$. The exact evaluations of (11*a*) and (11*b*) are

$$\begin{aligned} P(\mu, \tau) = {} & P_0\,\mathrm{e}^{-(\gamma\tau+\kappa\mu)}I_0(\theta) + P_0\,\mathrm{e}^{[(\alpha/\kappa)-\gamma]\tau}(1-J(\kappa\mu, \alpha\tau/\kappa)) \\ & + \gamma^{-1}E_0\,\mathrm{e}^{[(\alpha/\gamma)-\kappa]\mu}(1-J(\gamma\tau, \alpha\mu/\gamma)) \end{aligned} \tag{17a}$$

$$\begin{aligned} E(\mu, \tau) = {} & E_0\,\mathrm{e}^{-(\gamma\tau+\kappa\mu)}I_0(\theta) + E_0\,\mathrm{e}^{[(\alpha/\gamma)-\kappa]\mu}(1-J(\gamma\tau, \alpha\mu/\gamma)) \\ & + (\alpha/\kappa)P_0\,\mathrm{e}^{[(\alpha/\kappa)-\gamma]\tau}(1-J(\kappa\mu, \alpha\tau/\kappa)). \end{aligned} \tag{17b}$$

The function $J(u, v)$ is defined here as

$$J(u, v) = 1 - \mathrm{e}^{-v}\int_0^u \mathrm{e}^{-z}I_0(2\sqrt{vz})\,\mathrm{d}z \tag{18}$$

and has arisen in numerous physical contexts. Its properties have been studied extensively by Goldstein (1953) and many of them are presented in a convenient form by Luke (1962). A number of useful asymptotic expansions are available. The elementary properties of immediate interest are

$$J(u, 0) = \mathrm{e}^{-u} \tag{19a}$$

$$J(0, v) = 1 \tag{19b}$$

$$J(\infty, v) = 0 \tag{19c}$$

$$J(u, \infty) = 1 \tag{19d}$$

$$\partial J(u, v)/\partial u = -e^{-(u+v)} I_0(2\sqrt{uv}) \tag{19e}$$

$$\partial J(u, v)/\partial v = e^{-(u+v)} (u/v)^{1/2} I_1(2\sqrt{uv}). \tag{19f}$$

It may be verified, on using identities (19*a*)–(19*f*), that the original differential equations (8*a*) and (8*b*) are satisfied.

4. Specific physical applications

A number of conclusions concerning the types of physical behaviour represented by solutions of the MBE may be gleaned from the general linear solutions (11*a*) and (11*b*), as well as from the particular solutions (17*a*) and (17*b*). Firstly, one of the terms in the analytical expression for $E(\mu, \tau)$ is always proportional to α, so that in circumstances where this term is dominant we should expect superradiant behaviour (field intensities in superradiant processes are essentially proportional to the square of the number of active atoms, and α is proportional to this number). Furthermore, for large enough $\gamma\tau$ the linear solutions tend to values independent of τ, namely

$$P(\mu, \tau) \rightarrow P_{ss}(\mu) = \gamma^{-1} E_0 \exp\left[(\alpha/\gamma) - \kappa\right]\mu \tag{20a}$$

$$E(\mu, \tau) \rightarrow E_{ss}(\mu) = \gamma P_{ss}(\mu), \tag{20b}$$

which are certainly not superradiant. We can therefore conclude that superradiance is destroyed when homogeneous broadening is 'large enough'. The steady-state quantities $P_{ss}(\mu)$ and $E_{ss}(\mu)$ given by (20*a*) and (20*b*) are characteristic of ASE (Icsevgi and Lamb 1969, Hopf and Scully 1969, Crisp 1970, Allen and Peters 1973). Instead of the spatial distribution occurring when $\partial P/\partial\tau = 0$, let us now consider the temporal distribution which could conceivably occur when $\partial E/\partial\mu = 0$ in equations (4*a*) and (4*b*). Again keeping $N(\mu, \tau)$ equal to unity, we find that (4*a*) and (4*b*) are easily solved in this case, yielding

$$P(\tau) = P_0 \exp[(\alpha/\kappa) - \gamma]\tau \tag{21a}$$

$$E(\tau) = (\alpha/\kappa) P(\tau). \tag{21b}$$

These displaced-time-dependent quantities are recognised as describing the (linearised) phenomenon of SGA. The factor α/κ in equation (21*b*) indicates that superradiant behaviour is retained. Note also that the functions $P(\tau)$ and $E(\tau)$ are independent of L; thus, on transforming the quantities in (21*a*) into the original unscaled quantities we obtain for the argument of the exponential

$$(\beta c^{-1} k^{-1} - T_2^{-1})(t - x/c).$$

This independence from L is in keeping with the conclusions of previous authors (Bonifacio *et al* 1978) that there is no concept of cooperation length in SGA. It is also significant that the threshold condition for growth of the field and of the atomic polarisation in both ASE and SGA is $\alpha > \kappa\gamma$. In this connection it should be noted that recent work (Hermann 1979a,b) has shown that the condition $\alpha > \kappa\gamma$ is not sufficient to ensure that amplification will occur, and that a further constraint involving the amplitude and gain-to-loss ratio of the pumping pulse is also required. As the model we employ utilises a δ-pulse excitation interacting with an array of two-level atoms, it is unnecessary to investigate this further constraint here.

So far, we have not considered the requirement of providing precisely defined conditions, or constraints, which suffice to differentiate between the different physical processes of SF, ASE and SGA. In order to do so, we turn again to equations (17*a*) and (17*b*). A further property of the function $J(u, v)$ is required, namely that for $v > u$ there is a modified Bessel-function expansion (Luke 1962)

$$e^{v}(1-J(u, v)) = e^{-u} \sum_{m=1}^{\infty} (u/v)^{m/2} I_m(2\sqrt{uv}). \tag{22}$$

Thus, providing that $\alpha\mu > \gamma^2\tau$ and also $\alpha\tau > \kappa^2\mu$, we may expand (17*a*) and (17*b*) as

$$P(\mu, \tau) = e^{-(\gamma\tau+\kappa\mu)}\left(P_0 \sum_{m=0}^{\infty} (2\kappa\mu)^m \theta^{-m} I_m(\theta) + E_0 \sum_{m=1}^{\infty} \gamma^{m-1}(2\tau)^m \theta^{-m} I_m(\theta)\right) \tag{23a}$$

$$E(\mu, \tau) = e^{-(\gamma\tau+\kappa\mu)}\left(\alpha P_0 \sum_{m=1}^{\infty} \kappa^{m-1}(2\mu)^m \theta^{-m} I_m(\theta) + E_0 \sum_{m=0}^{\infty} (2\gamma\tau)^m \theta^{-m} I_m(\theta)\right). \tag{23b}$$

With $\gamma = \kappa = 0$, the right-hand side of equation (23*b*) becomes $2\alpha\mu P_0\theta^{-1}I_1(\theta) + E_0 I_0(\theta)$. We may regard the terms inside the large round brackets in (23*a*) and (23*b*) as 'gain' functions, and the exponential functions as 'loss' functions. According to equation (22), we find that for $\alpha\mu \gg \gamma^2\tau$ and $\alpha\tau \gg \kappa^2\mu$ the expansions may be truncated at the first modified Bessel function in each case, so that the 'gain' functions become independent of the loss parameters κ, γ. Let us also assume that $\theta \gg 1$; in the context of SF this implies that the delay time T_D of the leading pulse (which may be regarded as an upper limit of the displaced time $T = t - x/c$ in the linear regime) is much larger than the superradiant time T_R. Then the $I_m(\theta)$ may be replaced with their asymptotic limits for $\theta \to \infty$, i.e. $\exp(\theta - \frac{1}{2}\ln(2\pi\theta)) \sim \exp\theta$, and consequently $E(\mu, \tau) \sim (\alpha P_0 + E_0)\exp[\theta - (\gamma\tau + \kappa\mu)]$. The argument of the exponential function here is always positive, since we have assumed $\alpha\mu \gg \gamma^2\tau$, $\alpha\tau \gg \kappa^2\mu$. A further consequence of these two inequalities is that $\alpha \gg \kappa\mu$. We can conclude that superradiant emission in which the field always increases monotonically with θ (i.e. SF in the linear regime) will occur when the inequalities $2\alpha\mu P_0 \gg E_0$, $\alpha\mu \gg \gamma^2\tau$ and $\alpha\tau \gg \kappa^2\mu$ are satisfied. It is useful to express these inequalities in terms of easily observable quantities, and for this purpose we set $\mu = 1$, $\tau = \tau_D = (\frac{1}{2}T_E)^{-1}T_D$ and write $T_k = c^{-1}k^{-1}$, $T_c = \beta^{-1/2} = (\frac{1}{2}T_E T_R)^{1/2}$. The full set of constraints sufficient for SF is consequently found to be

$$T_D \gg T_R \gg T_c \gg \tfrac{1}{2}T_E \tag{24a}$$

$$T_2 \gg (T_R T_D)^{1/2} \tag{24b}$$

$$T_k \gg \tfrac{1}{2}T_E(T_R/T_D)^{1/2} \tag{24c}$$

$$\alpha P_0 \gg E_0. \tag{24d}$$

The time T_D is the delay to be expected of the first pulse in superfluorescent emission, and is given to a first approximation (Hermann 1979a, b, Hermann and Bullough 1979) by $T_D = \frac{1}{4}T_R\{\ln(4\sqrt{2\pi}/P_0)\}^2$. The inequality (24*b*) has already been given by Schuurmans and Polder (1979). Further constraints would be needed, of course, if additional dissipative and line-broadening processes (e.g. $T_2^* < \infty$) were incorporated in the equations of motion.

The constraints (24*a*) are necessary for SF, but are not generally required for amplification. For ASE it is easily established, from the properties of the functions $J(u, v)$, that sufficient conditions for obtaining the steady-state equations (20*a*) and (20*b*) are the inequality (24*c*) and the inequality $T_2 \ll (T_R T_D)^{1/2}$. Sufficient conditions for the onset of behaviour typical of SGA are likewise found to be (24*b*) together with the inequality $T_k \ll \frac{1}{2} T_E (T_R/T_D)^{1/2}$. Note that the latter condition implies that $\kappa \gg (T_D/T_R)^{1/2}$. In all of the cases we have considered the inequality $\alpha > \kappa\gamma$ holds, and is obviously to be regarded as the threshold condition for amplification generally. In terms of the unscaled parameters it becomes

$$(T_k T_2)^{1/2} > T_c. \quad (25)$$

5. Conclusion

In conclusion, it has been shown that an analytic solution of the semiclassical Maxwell–Bloch equations in an amplifying context is easily obtainable, in linear approximation, when field losses and atomic relaxation terms are included. The result should *not* be regarded as a trivial modification of the more restrictive calculations in which either one or both of these two types of dissipative processes are omitted: field losses are an essential ingredient in the physics of SGA, while atomic relaxation is essential for ASE. Hence, some understanding of the transitions from the regime of SF to the regimes of ASE or SGA can be obtained from the general analytic solution. Constraints sufficient for the purpose of defining the three regimes of SF, ASE and SGA have thus been obtained from the analytic solution. Note that Leonardi and Vaglica (1979) have recently performed a quantum-mechanical calculation of the process of SF emission incorporating inhomogeneous dephasing (characteristic time T_2^*). They have concluded that the constraint $(T_R T_D) < T_2^{*2}$ must be imposed for inhomogeneous broadening to have a negligible effect upon SF, and consequently that inhomogeneous dephasing does not contribute significantly to the dynamics of the original SF experiments of Skribanowitz *et al* (1973). This partly justifies our neglect of inhomogeneous broadening.

An extension of the present calculation to the *nonlinear* regime may be pursued along the lines indicated in the treatment of undamped SF by the present author (Hermann 1979a). The presence of damping terms, however, implies that a general nonlinear solution in terms of a single (similarity) variable is no longer possible. The details of this extension will be left to a future publication.

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